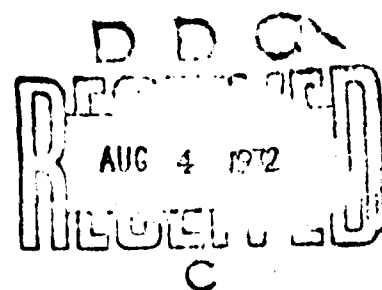AD 745958

AERO-ASTRONAUTICS REPORT NO. 94

COMPARISON OF SEVERAL GRADIENT ALGORITHMS
FOR MATHEMATICAL PROGRAMMING PROBLEMS

by

A. MIELE, J.L. TIETZE, and A.V. LEVY

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AUG 4 1972

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RICE UNIVERSITY

1972

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
Department of Mechanical and Aerospace Engineering and Materials Science Rice University, Houston, Texas 77001	UNCLASSIFIED
	2b GROUP

3 REPORT TITLE

COMPARISON OF SEVERAL GRADIENT ALGORITHMS FOR MATHEMATICAL PROGRAMMING PROBLEMS

4 DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Scientific   Interim

5 AUTHOR(S) *(First name, middle initial, last name)*

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6 REPORT DATE	7a. TOTAL NO OF PAGES	7b NO OF REFS
27 June 1972	29	3

8a. CONTRACT OR GRANT NO	9a ORIGINATOR'S REPORT NUMBER(S)
AFOSR-72-2185	
b. PROJECT NO	
9749	
c. 61102F	9b OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
	AFOSR - TR - 72-1530
d. 681304	

10 DISTRIBUTION STATEMENT

A.   Approved for public release; distribution unlimited.

11 SUPPLEMENTARY NOTES	12 SPONSORING MILITARY ACTIVITY
TECH, OTHER	Air Force Office of Scientific Research(NM) 1400 Wilson Blvd. Arlington, Virginia 22209

13 ABSTRACT

In this paper, the numerical solution of the basic problem of mathematical programming is considered. This is the problem of minimizing a function $f(x)$ subject to a constraint $\phi(x) = 0$. Here, f is a scalar, x an n-vector, and ϕ a q-vector, with $q < n$.

Six variations of the sequential gradient-restoration algorithm and the combined gradient-restoration algorithm are considered, and their relative efficiency (in terms of number of iterations for convergence) is evaluated. The variations being considered are as follows: (i)SGRA-CR, sequential gradient-restoration algorithm, complete restoration, (ii)SGRA-IR, sequential gradient-restoration algorithm, incomplete restoration, (iii)SGRA-OR, sequential gradient-restoration algorithm, optional restoration, (iv)CGRA-NR, combined gradient-restoration algorithm, no restoration, (v) CGRA-AR, combined gradient-restoration algorithm, alternate restoration, (vi)CGRA-OR, combined gradient-restoration algorithm, optional restoration

Evaluation of these algorithms is accomplished through eight numerical examples. The first two examples pertain to quadratic functions subject to linear constraints. The remaining examples pertain to nonquadratic functions subject to nonlinear constraints. The results indicate that (a)the inclusion of a restoration phase is necessary for rapid convergence and (b)the algorithms with alternate restoration or optional restoration are the most efficient among those considered here.

Comparison of Several Gradient Algorithms

For Mathematical Programming Problems[1]

by

A. MIELE[2], J.L. TIETZE[3], and A.V. LEVY[4]

Abstract. In this paper, the numerical solution of the basic problem of mathematical programming is considered. This is the problem of minimizing a function $f(x)$ subject to a constraint $\varphi(x) = 0$. Here, f is a scalar, x an n-vector, and φ a q-vector, with $q < n$.

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(i) SGRA-CR, sequential gradient-restoration algorithm, complete restoration,

(ii) SGRA-IR, sequential gradient-restoration algorithm, incomplete restoration,

(iii) SGRA-OR, sequential gradient-restoration algorithm, optional restoration,

(iv)   CGRA-NR, combined gradient-restoration algorithm, no restoration,

(v)   CGRA-AR, combined gradient-restoration algorithm, alternate restoration,

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1.   Introduction

In previous papers (Ref. 1-3), two basic algorithms for the minimization

of constrained functions were developed: the sequential gradient-restoration

algorithm (SGRA) and the combined gradient-restoration algorithm (CGRA). The

former is an iterative algorithm which consists of the alternate succession of

gradient phases and restoration phases; the latter is an iterative algorithm in

which the gradient phase and the restoration phase are combined in a single

phase.

In the gradient phase of SGRA, one generates a displacement Δx lowering

the value of the function, while avoiding excessive constraint violation; in the

restoration phase of SGRA, one generates a displacement Δx restoring the

constraint to a predetermined accuracy, while avoiding excessive change in the

value of the function. On the other hand, in the gradient-restoration phase of

CGRA, one generates a displacement Δx lowering the value of the augmented

function, while simultaneously reducing the constraint violation.

In this paper, six variations of the sequential gradient-restoration

algorithm and the combined gradient-restoration algorithm are considered, and

their relative efficiency (in terms of number of iterations for convergence) is

evaluated through eight numerical examples. The variations being considered

are indicated below:

(i)   SGRA-CR, sequential gradient-restoration algorithm, complete restoration,

(ii)   SGRA-IR, sequential gradient-restoration algorithm, incomplete restoration,

(iii) SGRA-OR, sequential gradient-restoration algorithm, optional restoration,

(iv)   CGRA-NR, combined gradient-restoration algorithm, no restoration,

(v)   CGRA-AR, combined gradient-restoration algorithm, alternate restoration,

(vi)   CGRA-OR, combined gradient-restoration algorithm, optional restoration.

2. Statement of the Problem

We consider the problem of minimizing the function

$$f = f(x) \tag{1}$$

subject to the constraint

$$\varphi(x) = 0 \tag{2}$$

where f is a scalar, x an n-vector, and φ a q-vector, with $q < n$. Here, all vectors are column vectors. It is assumed that the first and second partial derivatives of the function $f(x)$ and $\varphi(x)$ exist and are continuous and that the constrained minimum exists.

2.1. First-Order Conditions.

From theory of maxima and minima, it is known that the above problem is equivalent to that of minimizing the augmented function

$$F(x, \lambda) = f(x) + \lambda^T \varphi(x) \tag{3}$$

subject to the constraint (2). Here, the q-vector λ is the Lagrange multiplier and the superscript T denotes the transpose of a matrix. If

$$F_x(x, \lambda) = f_x(x) + \varphi_x(x)\lambda \tag{4}$$

denotes the gradient of the augmented function, the optimum solution for x and λ must satisfy the relations

$$\varphi(x) = 0 \quad , \quad F_x(x, \lambda) = 0 \tag{5}$$

which are a system of $n + q$ equations in the $n + q$ components of x and λ. In Eqs. (4)-(5), the gradients f_x and F_x denotes n-vectors and the matrix φ_x is $n \times q$.

2.2.   <u>Approximate Solutions</u>. Since the system (5) is generally nonlinear, approximate methods must be employed. In this connection, we introduce here the scalar performance indexes

$$P(x) = \varphi^T(x)\varphi(x) \quad , \quad Q(x, \lambda) = F_x^T(x, \lambda)F_x(x, \lambda) \tag{6}$$

which measure the errors in the constraint and the optimum condition, respectively. Then, we observe that $P = 0$ and $Q = 0$ for the optimum solution, while $P > 0$ and/or $Q > 0$ for any approximation to the solution. When approximate methods are used, they must ultimately lead to values of x and λ such that

$$P(x) \leq \epsilon_1 \quad , \quad Q(x, \lambda) \leq \epsilon_2 \tag{7}$$

Alternatively, (7) can be replaced by

$$R(x, \lambda) \leq \epsilon_3 \tag{8}$$

where

$$R(x, \lambda) = P(x) + Q(x, \lambda) \tag{9}$$

denotes the cumulative error in the constraint and the optimum condition. In (7)-(8), ϵ_1, ϵ_2, ϵ_3 are small, preselected numbers. Note that, if one chooses $\epsilon_1 = \epsilon_2 = \epsilon_3$, satisfaction of Ineq. (8) implies satisfaction of Ineqs. (7).

3.   Description of the Algorithms

In this section, the algorithms being investigated are described.

(i)   SGRA-CR: Sequential gradient-restoration algorithm, complete restoration. This algorithm consists of the alternate succession of gradient phases and restoration phases.

The gradient phase is started providing

$$P(x) \leq \epsilon_1 \tag{10}$$

It involves a single iteration, in which the augmented function is reduced subject to an upper limit for the constraint error, that is, [5]

$$F(\tilde{x}, \lambda) < F(x, \lambda) \quad , \quad P(\tilde{x}) \leq \epsilon_4 \tag{11}$$

The restoration phase is started providing

$$P(x) > \epsilon_1 \tag{12}$$

It involves several iterations, in each of which the constraint error is reduced, that is,

$$P(\tilde{x}) < P(x) \tag{13}$$

The restoration phases is terminated whenever Ineq. (10) is satisfied.

-----

[5] The symbol x denotes the nominal point, $\tilde{x}$ the varied point, and λ the Lagrange multiplier.

Remark. The algorithm is started with a gradient phase if Ineq. (10)

is satisfied or a restoration phase if Ineq. (10) is violated. Normally, a gradient

phase is followed by a restoration phase. Occasionally, the gradient phase is

followed by another gradient phase, that is, the restoration phase is bypassed:

this is precisely the case whenever Ineq. (10) is satisfied.

(ii)   SGRA-IR: Sequential gradient-restoration algorithm, incomplete

restoration. This algorithm consists of the alternate succession of gradient

phases and restoration phases.

The gradient phase is started regardless of whether Ineq. (10) is satisfied.

It involves a single iteration, in which the augmented function is reduced subject

to an upper limit on the constraint error, that is,

$$F(\tilde{x}, \lambda) < F(x, \lambda) \quad , \quad P(\tilde{x}) \le P(x) + \epsilon_4 \tag{14}$$

The restoration phase is started only if Ineq. (12) is satisfied. It

involves a single iteration, in which the constraint error is reduced in accordance

with Ineq. (13).

The starting condition and the bypassing condition for SGRA-IR are

identical with those of SGRA-CR (see Remark).

(iii)   SGRA-OR: Sequential gradient-restoration algorithm, optional

restoration. This algorithm consists of the alternate succession of gradient

phases and restoration phases.

The gradient phase is started providing

$$Z(x, \lambda) \le 1 \tag{15}$$

where the parameter Z is defined by

$$Z = \epsilon P(x)/Q(x, \lambda) \qquad (16)$$

with

$$\epsilon = \epsilon_2/\epsilon_1 \qquad (17)$$

It involves a single iteration, in which the augmented function is reduced in accordance with Ineqs. (14).

The restoration phase is started providing

$$Z(x, \lambda) > 1 \qquad (18)$$

It involves several iterations, in each of which the constraint error is reduced in accordance with Ineq. (13). The restoration phase is terminated whenever Ineq. (15) is satisfied.

The bypassing condition for SGRA-OR is identical with that of SGRA-CR (see Remark).

(iv)   CGRA-NR: Combined gradient-restoration algorithm, no restoration.
In this algorithm, the gradient phase and the restoration phase are combined
together in a single phase. It involves a single iteration, in which the augmented
function is reduced in accordance with Ineqs. (14).

(v)   CGRA-AR: Combined gradient-restoration algorithm, alternate
restoration. This algorithm consists of the alternate succession of combined
gradient-restoration phases and restoration phases.

The combined gradient-restoration phase is started regardless of whether
Ineq. (10) is satisfied. It involves a single iteration, in which the augmented
function is reduced in accordance with Ineqs. (14).

The restoration phase is started only if Ineq. (12) is satisfied. It
involves a single iteration, in which the constraint error is reduced in
accordance with Ineq. (13).

The starting condition and the bypassing condition for CGRA-AR are
identical with those of SGRA-CR (see Remark).

(vi)   CGRA-OR: Combined gradient-restoration algorithm, optional
restoration. This algorithm consists of the alternate succession of combined
gradient-restoration phases and restoration phases.

The combined gradient-restoration phase is started providing Ineq. (15)
is satisfied. It involves a single iteration, in which the augmented function is
reduced in accordance with Ineqs. (14).

The restoration phase is started providing Ineq. (18) is satisfied. It
involves several iterations, in each of which the constraint error is reduced

in accordance with Ineq. (13). The restoration phase is terminated whenever Ineq. (15) is satisfied.

The bypassing condition for CGRA-OR is identical with that of SGRA-CR (see Remark).

Remark. For the algorithms with optional restoration, the multiplier λ appearing in (15)-(18) is computed as follows. For SGRA-OR, Eq. (19-1) must be solved with $C_1 = 1$ and $C_2 = 0$. For CGRA-OR, Eq. (19-1) must be solved with $C_1 = 1$ and $C_2 = 1$.

4.   Generalized Algorithm

Let x denote the nominal point, $\tilde{x}$ the varied point, Δx the displacement leading from the nominal point to the varied point, a nd α the stepsize. With this understanding, the previous algorithms can be represented in the following generalized form:

$$\varphi_x^T(x)\varphi_x(x)\lambda + C_1\varphi_x^T(x)f_x(x) - C_2\varphi(x) = 0 \tag{19-1}$$

$$p = C_1 f_x(x) + \varphi_x(x)\lambda \tag{19-2}$$

$$\Delta x = -\alpha p \tag{19-3}$$

$$\tilde{x} = x + \Delta x \tag{19-4}$$

For given nominal point x and constants C_1 and C_2, Eqs. (19) constitute a complete iteration leading to the varied point $\tilde{x}$ providing one specifies the stepsize α. The constants C_1 and C_2 depend on the particular algorithm and take the values given in Table 1. The detailed derivation of Eqs. (19) is presented in Refs. 1-3 and, hence, is not repeated here.

Table 1. Characteristic constants.

Algorithm	Phase	C_1	C_2
SGRA	Gradient	1	0
	Restoration	0	1
CGRA	Gradient-restoration	1	1
	Restoration	0	1

5.   Stepsize Determination

For all of the previous algorithms, the position vector at the end of any step can be written as

$$\tilde{x} = x - \alpha p \tag{20}$$

where p denotes the search direction, which is given by (19-2). This is a one-parameter family of varied points $\tilde{x}$, for which the augmented function (3) , the constraint error (6-1), and the error in the optimum condition (6-2) take the form

$$F(\tilde{x}, \lambda) = F(x - \alpha p, \lambda) = \tilde{F}(\alpha) \tag{21}$$

$$P(\tilde{x}) = P(x - \alpha p) = \tilde{P}(\alpha) \tag{22}$$

$$Q(\tilde{x}, \lambda) = Q(x - \alpha p, \lambda) = \tilde{Q}(\alpha) \tag{23}$$

For the gradient phase of a SGRA-algorithm or the combined gradient-restoration phase of a CGRA-algorithm, Ineqs. (11) and (14) can be written in the general form

$$\tilde{F}(\alpha) < \tilde{F}(0) \quad , \quad \tilde{P}(\alpha) \leq \tilde{P}(0) + \epsilon_4 \tag{24}$$

Their satisfaction can be ensured by employing a bisection process, starting from a suitably chosen reference stepsize

$$\alpha = \alpha_o \tag{25}$$

For the determination of the reference stepsize, see Section 6.

For the restoration phase of a SGRA-algorithm or a CGRA-algorithm,

Ineq. (13) can be written as

$$\tilde{P}(\alpha) < \tilde{P}(0) \qquad (26)$$

Its satisfaction can be ensured by employing a bisection process, starting from

the reference stepsize

$$\alpha = 1 \qquad (27)$$

This value reduces the constraint error $P(x)$ to zero, if the constraint function

$\varphi(x)$ is linear in x.

6.   Reference Stepsize

The search technique outlined in Section 5 for the gradient stepsize

employs a bisection process, starting from the reference stepsize (25),

until satisfaction of Ineqs. (24) occurs. A procedure useful to determine this

reference stepsize is outlined here and is based on a quadratic representation

of the augmented function associated with the one-parameter family of solutions (20).

Let the function $\tilde{F}(\alpha)$ be represented in the quadratic form

$$\tilde{F}(\alpha) = k_0 + k_1\alpha + k_2\alpha^2 \tag{28}$$

and let the coefficients of the quadratic be determined so as to match the values

of the ordinate and the slope at $\alpha = 0$ and the value of the ordinate at $\alpha = 1$.

This yields the relations

$$\tilde{F}(0) = k_0 \quad , \quad \tilde{F}_\alpha(0) = k_1 \quad , \quad \tilde{F}(1) = k_0 + k_1 + k_2 \tag{29}$$

which imply that

$$k_0 = \tilde{F}(0) \quad , \quad k_1 = -\tilde{Q}(0) \quad , \quad k_2 = \tilde{F}(1) - \tilde{F}(0) + \tilde{Q}(0) \tag{30}$$

With the coefficients known, the following possibilities arise:

$$\text{(i)} \ \ k_2 > 0 \quad \text{or} \quad \text{(ii)} \ \ k_2 \leq 0 \tag{31}$$

In Case (i), the quadratic function (28) has a minimum for the following value

of the gradient stepsize:

$$\alpha = -k_1/2k_2 \tag{32}$$

In Case (ii), the quadratic function (28) decreases monotonically with α.

This suggests the use of the following reference values for the gradient stepsize:

$$\alpha_o \approx -k_1/2k_2 \quad \text{if} \quad k_2 > 0$$

$$\tag{33}$$

$$\alpha_o = 1 \qquad \text{if} \quad k_2 \leq 0$$

7. Experimental Conditions

In order to evaluate the previous algorithms, eight numerical examples were considered. The first two examples pertain to quadratic functions subject to linear constraints. The remaining examples pertain to nonquadratic functions subject to nonlinear constraints. Each example was solved with the three versions of SGRA and the three versions of CGRA outlined in Section 3. All of the algorithms were programmed in FORTRAN IV, and the numerical results were obtained using a Burroughs B-5500 computer and double-precision arithmetic.

Starting Point. For all of the examples, the nominal point chosen to start an algorithm was defined by

$$x_1 = x_2 = \ldots = x_n = 2 \tag{34}$$

w. re n denotes the dimension of the vector x.

Search Technique. The determination of the gradient stepsize and the restoration stepsize was performed in accordance with Sections 5 and 6. For the gradient phase, the stepsize α was subject to the inequalities

$$\tilde{F}(\alpha) < \tilde{F}(0) \quad , \quad \tilde{P}(\alpha) \le \tilde{P}(0) + 1 \tag{35}$$

For the restoration phase, the stepsize was subject to the inequality

$$\tilde{P}(\alpha) < \tilde{P}(0) \tag{36}$$

Convergence. Convergence of an algorithm was defined through the inequalities

$$P(x) \le 10^{-8} \quad , \quad Q(x, \lambda) \le 10^{-4} \tag{37}$$

Nonconvergence. Conversely, nonconvergence of an algorithm was
defined by means of the inequalities

$$\text{(a)} \qquad N > 100 \tag{38-1}$$

or

$$\text{(b)} \qquad N_s > 20 \tag{38-2}$$

or

$$\text{(c)} \qquad M > 0.4 \times 10^{69} \tag{38-3}$$

Here, N is the iteration number, N_s is the number of bisections of the stepsize α
required to satisfy Ineq. (35) or (36), and M is the modulus of any of the
quantities employed in the algorithm. Satisfaction of Ineq. (38-1) indicates
divergence or extreme slowness of convergence; satisfaction of Ineq. (38-2)
indicates extreme smallness of the displacement Δx; and satisfaction of Ineq. (38-3)
indicates exponential overflow. Each of these situations is undesirable.

8.   Numerical Examples

In this section, eight numerical examples are described. The first two examples pertain to quadratic functions subject to linear constraints. The remaining examples pertain to nonquadratic functions subject to nonlinear constraints.

Example 8.1. Consider the problem of minimizing the function

$$f = (x_1 - x_2)^2 + (x_2 + x_3 - 2)^2 + (x_4 - 1)^2 + (x_5 - 1)^2 \tag{39}$$

subject to the constraints

$$x_1 + 3x_2 = 0 \quad , \quad x_3 + x_4 - 2x_5 = 0 \quad , \quad x_2 - x_5 = 0 \tag{40}$$

This function admits the relative minimum $f = 4.0930$ at the point defined by

$$x_1 = -0.7674 \;\; , \;\; x_2 = 0.2558 \;\; , \;\; x_3 = 0.6279 \;\; , \;\; x_4 = -0.1162 \;\; , \;\; x_5 = 0.2558 \tag{41}$$

and

$$\lambda_1 = 2.0465 \;\; , \;\; \lambda_2 = 2.2325 \;\; , \;\; \lambda_3 = -5.9534 \tag{42}$$

Example 8.2. Consider the problem of minimizing the function

$$f = (4x_1 - x_2)^2 + (x_2 + x_3 - 2)^2 + (x_4 - 1)^2 + (x_5 - 1)^2 \tag{43}$$

subject to the constraints

$$x_1 + 3x_2 = 0 \quad , \quad x_3 + x_4 - 2x_5 = 0 \quad , \quad x_2 - x_5 = 0 \tag{44}$$

This function admits the relative minimum $f = 5.3266$ at the point defined by

$$x_1 = -0.9455 \times 10^{-1}, \ x_2 = 0.3151 \times 10^{-1}, \ x_3 = 0.5157, \ x_4 = -0.4527, \ x_5 = 0.3151 \times 10^{-1} \quad (45)$$

and

$$\lambda_1 = 3.2779 \ , \quad \lambda_2 = 2.9054 \ , \quad \lambda_3 = -7.7478 \quad\quad (46)$$

Example 8.3. Consider the problem of minimizing the function

$$f = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^4 \quad\quad (47)$$

subject to the constraint

$$x_1(1 + x_2^2) + x_3^4 - 4 - 3\sqrt{2} = 0 \quad\quad (48)$$

This function admits the relative minimum $f = 0.3256 \times 10^{-1}$ at the point defined by

$$x_1 = 1.1048 \ , \quad x_2 = 1.1966 \ , \quad x_3 = 1.5352 \quad\quad (49)$$

and

$$\lambda_1 = -0.1072 \times 10^{-1} \quad\quad (50)$$

Example 8.4. Consider the problem of minimizing the function

$$f = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_3 - 1)^2 + (x_4 - 1)^4 + (x_5 - 1)^6 \quad\quad (51)$$

subject to the constraints

$$x_1^2 x_4 + \sin(x_4 - x_5) - 2\sqrt{2} = 0 \ , \quad x_2 + x_3^4 x_4^2 - 8 - \sqrt{2} = 0 \quad\quad (52)$$

This function admits the relative minimum $f = 0.2415$ at the point defined by

$$x_1 = 1.1661 \ , \ \ x_2 = 1.1821 \ , \ \ x_3 = 1.3802 \ , \ \ x_4 = 1.5060 \ , \ \ x_5 = 0.6109 \qquad (53)$$

and

$$\lambda_1 = -0.8553 \times 10^{-1} \ , \ \ \lambda_2 = -0.3187 \times 10^{-1} \qquad (54)$$

Example 8.5. Consider the problem of minimizing the function

$$f = (x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^4 + (x_4 - x_5)^4 \qquad (55)$$

subject to the constraints

$$x_1 + x_2^2 + x_3^3 - 2 - 3\sqrt{2} = 0 \ , \ \ x_2 - x_3^2 + x_4 + 2 - 2\sqrt{2} = 0 \ , \ \ x_1 x_5 - 2 = 0 \qquad (56)$$

This function admits the relative minimum $f = 0.7877 \times 10^{-1}$ at the point defined by

$$x_1 = 1.1911 \ , \ \ x_2 = 1.3626 \ , \ \ x_3 = 1.4728 \ , \ \ x_4 = 1.6350 \ , \ \ x_5 = 1.6790 \qquad (57)$$

and

$$\lambda_1 = -0.3882 \times 10^{-1} \ , \ \ \lambda_2 = -0.1672 \times 10^{-1} \ , \ \ \lambda_3 = -0.2879 \times 10^{-3} \qquad (58)$$

Example 8.6. Consider the problem of minimizing the function

$$f = 0.01(x_1 - 1)^2 + (x_2 - x_1^2)^2 \qquad (59)$$

subject to the inequality constraint

$$x_1 \leq -1 \qquad (60)$$

Introduce the auxiliary variable x_3 defined by

$$x_1 + x_3^2 + 1 = 0 \tag{61}$$

Then, the previous problem can be recast as that of minimizing the function (59) subject to the equality constraint (61). The function (59) admits the relative minimum $f = 0.04$ at the point defined by

$$x_1 = -1 \quad, \quad x_2 = 1 \quad, \quad x_3 = 0 \tag{62}$$

and

$$\lambda_1 = 0.04 \tag{63}$$

Example 8.7. Consider the problem of minimizing the function

$$f = -x_1 \tag{64}$$

subject to the inequality constraints

$$x_2 \geq x_1^3 \quad, \quad x_2 \leq x_1^2 \tag{65}$$

Introduce the auxiliary variables x_3 and x_4 defined by

$$x_2 - x_1^3 - x_3^2 = 0 \quad, \quad x_1^2 - x_2 - x_4^2 = 0 \tag{66}$$

Then, the previous problem can be recast as that of minimizing the function (64) subject to the equality constraints (66). The function (64) admits the relative minimum $f = -1$ at the point defined by

$$x_1 = 1 \quad, \quad x_2 = 1 \quad, \quad x_3 = 0 \quad, \quad x_4 = 0 \tag{67}$$

and

$$\lambda_1 = -1 \quad , \quad \lambda_2 = -1 \tag{69}$$

Example 8.8. Consider the problem of minimizing the function

$$f = \log x_3 - x_2 \tag{69}$$

subject to the equality constraint

$$x_2^2 + x_3^2 - 4 = 0 \tag{70}$$

and the inequality constraint

$$x_3 \geq 1 \tag{71}$$

Introduce the auxiliary variable x_1 defined by

$$x_3 = 1 + x_1^2 \tag{72}$$

Then, the previous problem can be recast as that of minimizing the function

$$f = \log(1 + x_1^2) - x_2 \tag{73}$$

subject to the equality constraint

$$(1 + x_1^2)^2 + x_2^2 - 4 = 0 \tag{74}$$

Note that x_3 has been eliminated from the problem and can be computed a posteriori with (72). The function (73) admits the relative minimum $f = -\sqrt{3}$ at the point defined by

$$\lambda_2 = 0 \ , \quad x_2 = \sqrt{3} \ , \quad x_3 = 1 \tag{75}$$

and

$$\lambda_1 = 1/2\sqrt{3} \tag{76}$$

9.   Results and Conclusions

The examples described in Section 8 were solved with the three versions of SGRA and the three versions of CGRA described in Section 3. The numerical results are presented in Tables 2-3, where the number of iterations for convergence N_* is shown. For the eight examples considered, Table 4 shows the cumulative number of iterations for convergences ΣN_*. From the tables, the following conclusions arise: (a) a restoration of some form is necessary for rapid convergence; and (b) while SGRA-CR is the most stable among the algorithms considered here, rapidity of convergence can be increased somewhat if one employs algorithms with alternate restoration or optional restoration.

Table 2. Number of iterations for convergence N_*.

Example	SGRA-CR	SGRA-IR	SGRA-OR
8.1	5	5	5
8.2	8	8	8
8.3	18	14	16
8.4	56	51	42
8.5	8	7	7
8.6	15	12	16
8.7	9	15	9
8.8	11	11	10

Table 3. Number of iterations for convergence N_*.

Example	CGRA-NR	CGRA-AR	CGRA-OR
8.1	17	5	5
8.2	65	8	8
8.3	22	16	16
8.4	36	54	43
8.5	7	7	7
8.6	>100	19	13
8.7	13	7	9
8.8	15	8	10

References

1.   MIELE, A., HUANG, H.Y., and HEIDEMAN, J.C., Sequential Gradient-
     Restoration Algorithm for the Minimization of Constrained Functions,
     Ordinary and Conjugate Gradient Versions, Journal of Optimization Theory
     and Applications, Vol. 4, No. 4, 1969.

2.   MIELE, A., and HEIDEMAN, J.C., Mathematical Programming for Constrained
     Minimal Problems, Part 1, Sequential Gradient-Restoration Algorithm, Rice
     University, Aero-Astronautics Report No. 59, 1969.

3.   MIELE, A., HEIDEMAN, J.C., and LEVY, A.V., Mathematical Programming
     for Constrained Minimal Problems, Part 3, Combined Gradient-Restoration
     Algorithm, Rice University, Aero-Astronautics Report No. 68, 1970.

Table 4. Cumulative number of iterations for convergence ΣN_*.

Algorithm	ΣN_*
SGRA-CR	130
SGRA-IR	123
SGRA-OR	113
CGRA-NR	>275
CGRA-AR	124
CGRA-OR	111